The Optimal Research Contest: Subsidies or Prizes*

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Abstract

This paper investigates the optimal design of research contests. A "principal", who values an innovative technology, attempts to speed up the discovery. In order to minimize the expected amount of innovation time required, the principal decides how to allocate the fixed budget between a top-up prize (e.g. a procurement contract) and efficiency-enhancing subsidies (e.g. a research grant) to competing R&D firms. The study's main results are as follows. Firstly, the optimal contest preferentially subsidizes the ex ante less efficient firm. Secondly, more resources are devoted to research subsidies when the private benefit of the innovation to the successful innovator increases. Finally, more resources are allocated as subsidies when the innovation involves more uncertainty.

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1 Introduction

A contest is an institution in which economic agents are motivated to expend scarce resources by the prospect of winning a limited number of prizes. A wide range of competitive events can be viewed as contests. One salient example is an R&D contest, a type of contest that is frequently organized in order to procure new technology.[1] R&D contests have been widely observed to be efficient mechanisms for spurring innovative activities. As early as in 1714, the British Parliament offered a prize of £20,000 for a "practical and useful" means of determining longitude at sea (Che and Gale, 2003). In 1829, the directors of Liverpool and Manchester Railway set aside a winner-take-all premium of £500 for the designer of the most improved locomotive engine for the first ever passenger line (Day, 1971). The winning design propelled the world into the golden era of steam locomotion. The U.S. Department of Defense (DoD) has used prize incentives (e.g. military procurement contracts) extensively to stimulate research into defense technology. One recent event was the 2005 DARPA (Defense Advanced Research Projects Agency) Grand Challenge. This was a race among autonomous robots in the Mojave Desert along a 132.2 mile long route. There were 23 finalists and the winner, who was the first participant to complete the course, earned a $2 million prize. The long and non-exhaustive list of recognition prizes today is a testament to the popularity and importance of R&D contests in motivating innovations (see Konrad, 2007).

Besides prize incentives, financial subsidies or research grants make up another popular instrument used to stimulate scientific discovery (see Gonzáez, Jaumandreu and Pazo, 2005). There are numerous basic research projects that are sponsored by public funds in the United States, as well as in many other countries. In addition to the prizes that reward winners, the DoD often heavily subsidizes firms that participate in its research contests (Lichtenburg, 1967, and Che and Gale, 2003). It can be deduced through mere intuition that both a prize and a subsidy can provide competing parties with additional incentives to exert productive effort if they are properly administered. It remains unclear, however, to what extent the two appealing instruments could functionally either substitute or complement each other. This ambiguity naturally poses a challenge to the design of a research contest. In other words, the question is how to choose between these two instruments to foster certain innovation? This paper directly addresses this question by studying the optimal design of a research contest and the nature of these strategic instruments.

In this paper, a particular scenario is considered in which two R&D firms engage in scientific

[1] Other examples include influence politics, sports, college admissions and labor market competition within firms.

activities in order to produce a given innovation (e.g., the longitude contest held in 1714) sooner than the other.[2] A patent race framework, as suggested by Dasgupta and Stiglitz (1980), is adopted here to model the competition.[3,4] The firm that achieves this discovery first can patent the technology, while the other firm receives nothing. The winner enjoys a private benefit from this patent, which includes cash flow from royalties, procurement contracts from future buyers of the technology, revenues from other commercialization of this technology and reputation.

A third party is one that intends to speed up the innovation for its own benefit. This party is denoted through the use of the generic term "principal", which encompasses a wide variety of practical settings. The principal can be a firm that searches for a technical solution. For instance, Apple Inc. has frequently outsourced its research and manufacturing tasks to capable high-tech firms. The principal can also be a public agency that looks for a particular technology to fulfill its own goal. For instance, a country's Ministry of Health may demand an effective vaccine to rein in a deadly epidemic. Another example is that DoD actively seeks reliable combative robots to fight off snipers in Baghdad. Alternatively, the principal can be a non-profit science foundation dedicated to inspiring scientific breakthroughs, such as resolving a major mathematical puzzle.

The principal attempts to speed up this discovery by utilizing its limited financial resources. The expected innovation time is jointly determined by the outputs of these firms. The technological output of a firm depends on both its research capacity or efficiency (the quality of laboratory equipment and scientists) and its subsequent input (autonomous research effort). Subject to a fixed budget, the principal has the flexibility to either promise a top-up prize (e.g., an additional procurement contract) for the winner in order to step up the firms' inputs, or provide subsidies (e.g., research grants) to the firms in order to improve their efficiency.[5]

This paper explicitly assumes that the principal utilizes its resources solely to minimize the expected innovation time. Gradstein and Konrad (1999) argue that the structure or rules of a contest are usually deliberately designed by the contest's organizer, and "result from the careful consideration of a variety of objectives". Although a contest can carry out various missions and use differing performance measures, a common concern when designing contests is the efficient

[2] Taylor (1995), Fullerton and McAfee (1999) and Che and Gale (2003) have shown why it is optimal to shortlist only two contestants in different R&D contest settings.

[3] As pointed out by Baye and Hoppe (2003) and Fu and Lu (2007b), this type of patent race model is of stochastic equivalence to a Tullock contest.

[4] This model has been popularly adopted in the literature. For instance, Vincenzo Denicolò (2000) adopts this model in a study on optimal patent policy.

[5] The research grants can be used by the recipients to buy new instruments and hire more researchers.

provision of appropriate incentives to elicit the desired amount of effort.

Much research has been carried out to examine how contests can be better designed to maximize overall effort.[6] At the same time, studies on research tournaments, where firms compete on the quality of their innovative products, have focused on an alternative objective: the value of the winning technology (Taylor, 1995, Fullerton and McAfee, 1999, and Che and Gale, 2003). Obviously, this outcome is positively related to the amount of effort exerted by each contestant. In our setting, where contestants compete to develop an innovation of a given nature, a natural goal of the principal is to shorten the delivery cycle.[7] The length of the delivery cycle is jointly determined by the effort profiles of the competing firms, as well as the nature of the innovation project and their research capacities.

While a more generous prize purse unambiguously contributes to a quicker discovery as it encourages firms to step up their efforts, more research subsidies could affect the innovation time through multiple venues. On the one hand, they can directly speed up the innovation, as subsidies enhance firms' efficiencies and amplify the technological outputs of the firms for given inputs. On the other hand, the improved efficiencies rendered by the research subsidies alter firms' incentives to exert research effort, while the nature and magnitude of this indirect effect has yet to be identified. Given the fixed budget of the principal, however, additional research subsidies crowd out the resource that is otherwise available for a more generous prize purse. These competing forces thus blur the trade-off between the two options.

Our paper shows that although both subsidies and prize incentives facilitate success, their functions differ subtly and the ability of one to substitute the other is limited. In particular, it has been found that the optimal budget allocation depends critically on the subtle interactions among three main factors: the existing value of the patent (private benefit) to the successful innovator, the technological nature of this innovation and the initial technological endowments (research capacity) of the firms.

Firstly, innovations vary in the amount of private benefit that could be gained by the winner. For instance, an applied research project could provide considerable commercial value, while basic research usually does not. A civil technology can be widely licensed to reap immediate rewards, while military technology may not. A scientist in a developing country often expects little returns from his success due to weak intellectual property rights protection. A lower patent value thus

[6] See Konrad (2007) for a thorough survey of theoretical work on contests.

[7] Dasgupta and Stiglitz (1980) discuss the factors that shorten innovation time.

propels the principal to compensate for this missing incentive by enriching the prize purse. We find that when the private benefit to the innovator is sufficiently low, neither firm is subsidized and all the money is added to a prize. By way of contrast, when the patent itself has a substantial value, an additional prize incentive would provide less incentive to motivate the firms to exert additional effort. Instead, in such cases, a greater share of research support in the form of subsidies would appear to reduce the expected innovation time.[8]

Secondly, innovative activities differ in their technological nature. In the setting discussed in this paper, the technological nature of an innovation specifically refers to the elasticity of the hazard rate (the conditional likelihood of discovery) to a firm's effort supply. That is, the higher the elasticity, the more a given amount of incremental effort can contribute to success, and the less uncertainty or difficulty is involved. In contrast, when the hazard rate is less responsive to additional effort, the discovery being pursued requires a greater amount of sacrifice. For example, additional research input could significantly expedite the development of a diet drug, while it would do less for the development of an effective HIV vaccine. We unanimously find that higher elasticity leads to lower subsidies but a more generous prize purse. Higher elasticity and additional research subsidies function analogously: both (directly) contribute to the likelihood of success for any given effort input. Higher elasticity is therefore a substitute for research subsidies and redirects resources into the prize purse in the optimum. This result provides important insights into the design of incentive mechanisms that facilitate innovation: prize incentives will sufficiently motivate targeted research only if the project of interest involves a moderate level of difficulty, while subsidies provide stronger motives when competing parties hold out weak prospects of success.

Thirdly, the most intricate strategic interactions are triggered when the principal distributes subsidies between competing firms. Firms may differ in their initial capabilities due to heterogeneity in terms of their physical capacity, human capital and knowledge stock.[9] Although a subsidy may provide a catalyst for success for any given effort input, its impact on equilibrium effort supplies is mixed. On the one hand, it encourages the recipient to work harder, as a subsidy is qualitatively equivalent to a reduction in marginal cost of effort. On the other hand, it could either level or

[8]Enhanced research capacities can induce contestants to exert a greater amount of research effort. This has been empirically supported (e.g. Lach (2002) among others).

[9]The heterogenity of firms can be attributed to their financial liquidity as well. Financial constraint that restricts a firm's action space is not explicitly included here. However, financial constraint (on effort supply) can be technically interpreted as a low efficiency parameter in our model. More details will be revealed by the model and are discussed in Section 4.

imbalance the playing field indefinitely, which alters competing firms' incentive to engage in effort, and, in turn, set off additional strategic interactions between them. It is shown here that a firm must be more preferentially subsidized when it is initially less efficient than the other, unless the patent carries little existing value (private benefit).[10] A subsidy may efficiently shorten the cycle only if it fills in the gap in firms' initial endowments of technology, while it provides a negative incentive (i.e., reduces equilibrium effort supply) when it exacerbates initial asymmetry.

This paper adds to the literature on the optimal design of the innovation race.[11] One obvious contribution of this paper is that it explicitly includes financial subsidies in the portfolio of strategic instruments. A number of studies have concerned themselves with optimally allocating prizes of differing ranks in innovation races as well as other type of contests. For instance, Denicolò and Franzoni (2007) investigate the optimal use of the winner-take-all principle in innovation races under different market conditions.[12] To the best of our knowledge, one important dimension of analysis has been ignored thus far, which is that the organizer may allocate his budget to improve participants' capabilities besides enriching the winner's purse.[13]

This paper is also closely linked to the literature on optimal handicap. Conventional wisdom teaches that a more level playing field creates more competition. A handful of papers have analytically implemented this logic in different contexts. For instance, Che and Gale (2003) show that in a research contest where firms compete based on the quality of the technology, imposing a bidding cap encourages both contestants to step up effort supplies. Unlike Che and Gale (2003), however, this paper studies research contests where firms race towards an innovation of a given nature, and the mechanism derived in this paper positively assists the weaker instead of handicapping the stronger.[14] Both approaches aim to balance the competition, and this paper complements Che and Gale's (2003) in this regard.

The remainder of this paper is structured as follows. Section 2 presents the setup of the model. Section 3 executes the formal analysis on optimal contest design. Section 4 discusses the results.

[10] In that case, no subsidy would be provided in the equilibrium.

[11] See Konrad (2007) for a thorough survey of theoretical works on contests.

[12] Other such studies include Moldovanu and Sela (2001), Matros (2005), Rosen, (1986), and Fu and Lu (2007a).

[13] The contestants may be self-motivated to make pre-contest investments to improve their fighting ability. This case has been analyzed by Kräkel (2004), Münster (2007) and Fu and Lu (2008), among others.

[14] Che and Gale (1998) studied the effect of a contribution cap on influence politics, and showed that it intensifies the competition. The role of a bidding cap has also been discussed by Fang (2002). Similarly, Fu (2006) has shown that in an all-pay auction with heterogenous players, total effort and the expected winning bid can be simultaneously and uniquely maximized when the weak contestant's bid is properly scaled up.

Concluding remarks are made in Section 5.

2 The Model

Two R&D firms, indexed by $i = 1, 2$, are engaged in a race towards an innovative technology. The firm that succeeds first could patent this technology and secures a private benefit $V_0 \geq 0$. A third party (e.g. a public agency like DoD) benefits from this innovation, and hence is eager to obtain this technology. This party is generically named a "principal". The principal has a total budget of M to foster this innovation. Specifically, the principal intends to spend its money eliciting the innovative effort supplied by the two competing R&D firms to minimize the expected innovation time. In order to fulfill this objective, the principal has to optimally divide its budget M into two parts: direct subsidies to firms (S_1 and S_2) and a top-up winning prize (Γ_0), e.g. a procurement contract, with $S_1, S_2, \Gamma_0 \geq 0$.

Each firm i invests an R&D effort of x_i on this research project in order to achieve a quicker discovery. The first successful innovator receives a total winner's purse of $V_0+\Gamma_0$ as its reward, while the other firm gets nothing. For the sake of tractability, the framework of Dasgupta and Stiglitz (1980) is adopted here to model this R&D race.[15] The actual time t_i for firm i to accomplish this task is a random variable that follows a Weibull (minimum) distribution. To put it formally, given x_i, the probability that firm i successfully innovates before time t is given by

$$F_i(t|x_i) = 1 - e^{-h_i(x_i)t},\ t \geq 0,\ i = 1, 2, \tag{1}$$

where $h_i(x_i)$ is firm i's hazard rate of success, i.e., firm i's conditional probability of making the discovery between time t and time $t + \Delta t$, provided that the discovery has not been achieved before time t. It is assumed that $h_i(x_i)$ is strictly increasing with effort x_i, and is concave in its argument. Clearly, $h_i(x_i)$ measures the technical output of firm i's innovation activities. Specifically, it is assumed, for analytical tractability, that the hazard rate $h_i(x_i)$ takes the functional form $h_i(x_i) = k_i x_i^r$, with $k_i > 0$ and $r \in (0, 1]$. The parameter r, which measures the elasticity of the hazard

[15] This approach assumes that each competing firm commits to a one-shot lump-sum R&D outlay that determines the time distribution of its success. Denicolò (2000) and Baye and Hoppe (2003), among many others, have followed this approaches in modelling innovation races. Kaplan, Luski and Wettsein (2003) suggest an approach that is analogous to an all-pay auction but also requires lump-sum effort. Another popular approach has been proposed by Lee and Wilde (1980) that requires each competing firm to commit to its R&D investment rate. Please refer to Reinganum (1989) for a thorough discussion on these modelling approaches. A recent application of Lee and Wilde (1980)'s model can be seen in Etro (2004).

rate to additional effort, is common to both firms. Determined by the technological nature of this innovation, it indicates the effectiveness of the R&D efforts in this particular innovation activity.[16] A smaller r depicts an innovation that involves more uncertainty. Parameter k_i reflects the impact of research capacity (e.g., quality of laboratory instruments and scientists) other than the researcher' autonomous efforts on the innovation efficiency, i.e., the efficiency or capability of a firm. The two firms could differ ex ante in their existing capacities, and a subsidy from the principal directly improves their capacities. It is assumed that $k_i \equiv \theta_i + S_i$, where θ_i indicates the firm's initial innovative capability, while S_i denotes the research subsidy the firm receives from the principal. As a result, a firm i's conditional probability of making the discovery at the instant time t is given by

$$h_i(x_i) = (\theta_i + S_i)x_i^r. \tag{2}$$

This setting thus intuitively captures the notion that a research grant amplifies a firm's R&D output. For example, the firm could spend the grant upgrading laboratory equipment and hiring additional scientists. Additional physical or human capital stock boosts the firm's productivity, as it allows the firm to conduct more parallel experiments. Without a loss of generality, it is assumed that $\theta_1 \geq \theta_2$, which indicates that Firm 1 has an ex ante larger research capacity than Firm 2. It is assumed throughout this paper that the R&D firms are subjected to limited liability, which thus requires that the monetary transfer S_i from the principal be non-negative.

Hence, from the viewpoint of the principal, for a given effort profile (x_1, x_2), the innovation time has a cumulative distribution function

$$\begin{aligned} F\left(t|x_1, x_2\right) &= 1 - \left(1 - F_1(t|x_1)\right)\left(1 - F_2(t|x_2)\right) \\ &= 1 - e^{-[h_1(x_1)+h_2(x_2)]t}, \;\; t \geq 0. \end{aligned} \tag{3}$$

The expected innovation time is then given by

$$E\left(t|x_1, x_2\right) = \frac{1}{h_1(x_1) + h_2(x_2)}. \tag{4}$$

We assume that the principal intends to minimize the expected innovation time as given by (4). The principal has to determine the optimal research grant profile $(S_1,\, S_2)$ that subsidizes the competing firms, and a top-up prize Γ_0 that rewards the winner. She is subject to the following budget constraint:

$$S_1 + S_2 + \;\Gamma_0 \leq M,\; S_1 \geq 0,\; S_2 \geq 0,\;\; \Gamma_0 \geq 0. \tag{5}$$

[16] See Fu and Lu (2007b) for further interpretations of parameter r. Fu and Lu (2007b) have shown that a patent race model is equivalent to a Tullock contest with impact functions of $h_i(\cdot)$.

Formally, a two-stage game is considered. In the first stage, the principal announces the rule of the contest, which is represented by the profile (S_1, S_2, Γ_0). It strategically sets the amounts of research grants and the top-up prize to minimize the expected innovation time $E\left(t|x_1, x_2\right)$. In the second stage, firms simultaneously commit to their R&D outlays x_1 and x_2 in order to maximize their expected payoffs.

It is assumed that R&D effort incurs a unitary marginal cost to each firm, i.e., $C_i(x_i) = x_i$. Hence, a firm i's expected payoff is given by

$$\pi_i(x_i, x_j) = \Pr\left(t_i < t_j|x_i, x_j\right)V - x_i,\ i = 1, 2,$$

where $V \equiv V_0 + \Gamma_0$ is the total reward received by the winner, which includes the private benefit from the patent V_0 and the value of the top-up prize Γ_0 awarded by the principal. This paper attempts to solve for the subgame perfect Nash equilibrium. That is, we intend to find out the optimally designed contest $(S_1^*, S_2^*, \Gamma_0^*)$ announced by the principal, when it anticipates the equilibrium responses of the competing firms $(x_1(S_1^*, S_2^*, \Gamma_0^*), x_2(S_1^*, S_2^*, \Gamma_0^*))$.

An Alternative Derivation of the Principal's Objective

It has explicitly been assumed that the principal solely minimizes the expected innovation time. This objective can be alternatively motivated by the following derivation. Assume that the principal receives a benefit B for each unit of time from consuming the innovation, and that it has a time discount rate of ρ. A rational principal would maximize its expected payoff as follows.

$$\begin{aligned} & U(h_1\left(x_1\right), h_2\left(x_2\right), \rho) \\ = & \int_0^\infty \left(\int_t^\infty e^{-\rho s} B ds\right)\left(h_1\left(x_1\right) + h_2\left(x_2\right)\right) e^{-(h_1(x_1)+h_2(x_2))t} dt \\ = & \frac{B}{\rho}[1 - \frac{\rho}{h_1\left(x_1\right) + h_2\left(x_2\right) + \rho}]. \end{aligned} \tag{6}$$

Thus, the principal maximizing (6) in fact minimizes the expected innovation time as given by (4). Both objectives are equivalent to maximizing the aggregate technological output $[h_1(x_1) + h_2(x_2)]$. For the sake of analytical tractability, it is assumed that the firms do not discount future payoffs. This setting, however, corresponds with a situation in which early innovation is more appealing to the principal than to the innovators.[17] For instance, an effective vaccine could create substantial

[17] Please refer to Section 4.5 for a more detailed discussion of this.

social benefits in addition to the private benefit that is received by the innovative pharmaceutical company.

3 The Analysis

This game is solved by backward induction. Each firm's equilibrium effort outlay in every given subgame is first characterized. The optimal allocation of the budget is then searched out that minimizes the expected innovation time.

3.1 Equilibrium R&D Effort

A firm wins if it realizes the desired discovery sooner than the other. The winning probability of a firm i is then given by

$$\begin{aligned}\Pr\left(t_i < t_j | x_i, x_j\right) &= \int_0^\infty \left(\int_0^{t_j} F_i'\left(t_i|x_i\right) dt_i\right) F_j'\left(t_j|x_j\right) dt_j \\ &= \frac{h_i(x_i)}{h_i(x_i) + h_j(x_j)}.\end{aligned}$$

Recall that the hazard rate $h_i(x_i) = k_i x_i^r = (\theta_i + S_i)x_i^r$. Firm i's expected payoff is thus rewritten as

$$\begin{aligned}\pi_i(x_i, x_j) &= \Pr\left(t_i < t_j | x_i, x_j\right) V - x_i \\ &= \frac{(\theta_i + S_i)x_i^r}{(\theta_i + S_i)\, x_i^r + (\theta_j + S_j)\, x_j^r} (V_0 + \Gamma_0) - x_i,\ i = 1, 2.\end{aligned}$$

The first-order condition for the firm i's expected payoff maximization problem is given by

$$\frac{\partial \pi_i}{\partial x_i} = \frac{r x_i^{r-1} x_j^r (\theta_i + S_i)\,(\theta_j + S_j)}{[(\theta_i + S_i)\, x_i^r + (\theta_j + S_j)\, x_j^r]^2} (V_0 + \Gamma_0) - 1 = 0,\ i = 1, 2.$$

The equilibrium effort for a given allocation profile (S_1, S_2, Γ_0) is thus obtained as follows

$$x_1^* = x_2^* = x^* = \frac{r(\theta_1 + S_1)\,(\theta_2 + S_2)}{[(\theta_1 + S_1) + (\theta_2 + S_2)]^2} (V_0 + \Gamma_0)\,. \tag{7}$$

Notably, the two firms invest the same amount of R&D effort in the equilibrium, regardless of the levels of their ex ante research capacities.

3.2 The Optimal Budget Allocation

Having obtained the contestants' equilibrium effort outlays in any given contest (S_1, S_2, Γ_0), the principal's optimal budget allocation problem is now probed. The principal's objective is to minimize the expected innovation time. Given the equilibrium effort function (7), the expected innovation time is thus written as

$$E(t) = \frac{1}{[(\theta_1 + S_1) + (\theta_2 + S_2)]x^{*r}} = \frac{[(\theta_1 + S_1) + (\theta_2 + S_2)]^{2r-1}}{[r(\theta_1 + S_1)(\theta_2 + S_2)(V_0 + \Gamma_0)]^r}. \tag{8}$$

The principal is to set the optimal bundle $(S_1^*, S_2^*, \Gamma_0^*)$ to minimize (8), subject to constraints (5). The principal allocates the resources among the three elements by comparing their marginal impacts on $E(t)$, which are given by

$$\frac{\partial E(t)}{\partial S_1} = E(t)\left[\frac{2r-1}{\theta_1 + \theta_2 + S_1 + S_2} - \frac{r}{\theta_1 + S_1}\right], \tag{9}$$

$$\frac{\partial E(t)}{\partial S_2} = E(t)\left[\frac{2r-1}{\theta_1 + \theta_2 + S_1 + S_2} - \frac{r}{\theta_2 + S_2}\right], \tag{10}$$

and

$$\frac{\partial E(t)}{\partial \Gamma_0} = E(t)\frac{-r}{V_0 + \Gamma_0}. \tag{11}$$

Before formally solving for the optimal budget allocation plan $(S_1^*, S_2^*, \Gamma_0^*)$, we first examine the role played by these strategic instruments. For descriptive convenience, we define $\xi_1 = \frac{r}{\theta_1+S_1} - \frac{2r-1}{\theta_1+\theta_2+S_1+S_2} > 0$, $\xi_2 = \frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+\theta_2+S_1+S_2} > 0$ and $\xi_3 = \frac{r}{V_0+\Gamma_0} > 0$. Note that the optimal allocation of the budget solely depends on comparisons among ξ_1, ξ_2 and ξ_3.

Lemma 1 *Let ς denote any of the three instruments (S_1, S_2, Γ_0). $\frac{\partial E(t)}{\partial \varsigma} < 0$ and $\frac{\partial^2 E(t)}{\partial^2 \varsigma} > 0$, regardless of the existing allocation (S_1, S_2, Γ_0).*

Proof. See Appendix. ■

Lemma 1 manifests the constantly positive effects of the three strategic instruments (S_1, S_2, Γ_0) on quicker success. Regardless of the existing allocation (S_1, S_2, Γ_0), a more generous prize purse or additional subsidy to either recipient always helps reduce $E(t)$. However, the marginal impact of an instrument strictly decreases as the amount of resources allocated to the instrument increases. As a result of Lemma 1, the optimal allocation profile requires the budget of the principal to be binding.

We next examine how these strategic instruments affect firms' incentives of effort supply. Obviously, a firm's equilibrium effort strictly increases with the top-up prize Γ_0. Thus, a greater top-up prize always help reduce the expected innovation time. However, the impact of an additional subsidy on a firm's incentive to supply effort is subtly ambiguous. Take the first order partial derivative of x_i^* with respect to S_i and we obtain

$$\frac{\partial x_i^*}{\partial S_i} = \frac{r\left(V_0 + \Gamma_0\right)\left(\theta_j + S_j\right)}{\left[\left(\theta_i + S_i\right) + \left(\theta_j + S_j\right)\right]^3} \cdot \left[\left(\theta_j + S_j\right) - \left(\theta_i + S_i\right)\right]. \tag{12}$$

The sign of $\frac{\partial x_i^*}{\partial S_i}$ thus depends on $(\theta_j + S_j) - (\theta_i + S_i)$. It reveals that providing an additional subsidy to a firm could stimulate its equilibrium effort supply if and only if the recipient is weaker than the other in terms of research capacity.

Furthermore, taking the first order partial derivative of x_i^* with S_j yields

$$\frac{\partial x_i^*}{\partial S_j} = \frac{r\left(V_0 + \Gamma_0\right)\left(\theta_i + S_i\right)}{\left[\left(\theta_i + S_i\right) + \left(\theta_j + S_j\right)\right]^3} \cdot \left[\left(\theta_i + S_i\right) - \left(\theta_j + S_j\right)\right], \tag{13}$$

which implies that when its rival receives an additional subsidy, a firm would increase its effort supply if and only if it is stronger.

These observations therefore lead to the following preliminary result that tremendously eases the optimal design problem is shown.

Lemma 2 *The optimal budget allocation must satisfy the following conditions:*

(a) Firm 1 remains at least as strong as Firm 2, i.e., $\theta_1 + S_1^* \geq \theta_2 + S_2^*$;

(b) If Firm 1 receives a positive subsidy, then Firm 2 must be as strong as Firm 1, i.e., $\theta_1 + S_1^* = \theta_2 + S_2^*$ *if* $S_1^* > 0$;

(c) If Firm 1 remains strictly stronger than Firm 2, then it must receive no subsidy, i.e., $S_1^* = 0$ *if* $\theta_1 + S_1^* > \theta_2 + S_2^*$.

Proof. Lemma 2 concerns itself with the allocation of subsidies between the two firms. We imagine that an infinitesimal amount of additional money is to be given to firm i. Since $E\left(t\right) = \frac{1}{((\theta_1+S_1)+(\theta_2+S_2))x^{*r}}$, its marginal impact on $E(t)$ is given by

$$\frac{dE(t)}{dS_i} = \frac{\partial E(t)}{\partial S_i} + \frac{\partial E(t)}{\partial x^*} \cdot \frac{\partial x^*}{\partial S_i}, \tag{14}$$

which is a composition of a direct effect

$$\frac{\partial E(t)}{\partial S_i} = \frac{-x^{*r}}{\{[(\theta_i + S_i) + (\theta_j + S_j)]x^{*r}\}^2}, \tag{15}$$

and an indirect effect $\frac{\partial E(t)}{\partial x^*} \cdot \frac{\partial x^*}{\partial S_i}$.

One observes from (15) that the direct effect of $S_i, i = 1, 2$ always has a negative sign, i.e., the direct effect tends to reduce the expected innovation time, regardless of the identity of the recipient. In addition, the magnitude of this first order impact does not depend on the identity of the recipient of subsidy. However, the sign of the indirect effect is indefinite. Apparently, $E(t)$ strictly decreases with equilibrium effort x^*. By (12) and (13), when the stronger (weaker) efficient firm is further subsidized, the equilibrium effort decreases (increases). This implies that when an additional amount of money is available to subsidize firms, it is a strictly dominant strategy to allocate it to the currently weaker firm to reduce the innovation time. We therefore conclude that $\theta_1 + S_1^* \geq \theta_2 + S_2^*$ must hold in the equilibrium. That is, the optimal allocation plan cannot reverse the initial asymmetry, which gives rise to Lemma 1(a). Results (b) and (c) immediately follow.

Q.E.D. ■

A subsidy to a firm affects the expected innovation through two avenues. Recall that the expected innovation time is given by $E(t) = \frac{1}{(k_i+k_j)x^{*r}}$. On the one hand, an additional subsidy to a firm tends to reduce the innovation time, as it improves the firm's performance for any given profile of effort outlays. On the other hand, it could indirectly affect the innovation time as it varies the equilibrium effort x^*. The former effect is obviously negative (i.e., it reduces the expected innovation time). By way of contrast, the direction of the indirect effect critically depends on the identity of the recipient. Its nature is directly revealed by (12) and (13). These equations unambiguously indicate that additional funds that subsidize a firm increases both firms' incentive to supply effort if and only if the recipient is strictly weaker than the other. This result thus demonstrates the subtle role played by a subsidy: it spurs on additional competition when it levels the playing field, while stifling the competition when it exacerbates existing imbalance.

As the amount of time for innovation strictly decreases in the effort exerted by each firm, the implication of the above discussions for the optimal contest design becomes clear. In order to maximally reduce the amount of time that firms require to achieve the innovation with a limited amount of resources, the optimal allocation bundle would preferentially subsidize the ex ante weaker firm, while it never allows this firm to ex post leapfrog the other.

Hence, in the subsequent analysis, two possible cases are considered: (1) $\theta_1 - \theta_2 > M$ and (2) $\theta_1 - \theta_2 \leq M$. In the former case, severe capacity asymmetry exists across the firms, and the resource available to the principal does not suffice to fill in the gap. Consequently, Firm 1 would remain strictly more efficient than Firm 2 regardless of the allocation plan. In the latter case, the initial asymmetry between the two firms is relatively mild and the principal could fully balance the competition using its budget, although a fully symmetric confrontation may not be optimal.

3.2.1 Case I: Severe Asymmetry ($\theta_1 - \theta_2 > M$)

The following preliminary result on the distribution of a research grant between the two firms is first established.

Lemma 3 *When $\theta_1 - \theta_2 > M$, the optimally designed contest does not subsidize Firm 1, i.e., $S_1^* = 0$.*

Lemma 3 directly follows from Lemma 2(c). When the funds available to the principal are insufficient to fully balance the playing field, any subsidy to Firm 1 provides only a negative incentive to the firms as it further upsets the balance. Hence, three possible optimal allocation plans could result: (1) $S_2^* = M$, and $\Gamma_0^* = 0$, (2) S_2^*, $\Gamma_0^* > 0$, $S_2^* + \Gamma_0^* = M$ and (3) $S_2^* = 0$, $\Gamma_0^* = M$.

In view of the fact that Firm 1 never receives any subsidy in this case, (8) leads to

$$E\left(t\right) = \frac{\left[\theta_1 + \left(\theta_2 + S_2\right)\right]^{2r-1}}{\left[r\theta_1\left(\theta_2 + S_2\right)\left(V_0 + \Gamma_0\right)\right]^r}. \tag{16}$$

The principal minimizes (16) subject to the constraint $S_2 + \Gamma_0 = M$, $S_2 \geq 0$, $\Gamma_0 \geq 0$. (16) leads to

$$\frac{\partial E\left(t\right)}{\partial S_2} = E(t) \cdot \left[\frac{2r-1}{\theta_1 + \left(\theta_2 + S_2\right)} - \frac{r}{\theta_2 + S_2}\right]; \tag{17}$$

$$\text{and } \frac{\partial E\left(t\right)}{\partial \Gamma_0} = E(t) \cdot \frac{-r}{V_0 + \Gamma_0}. \tag{18}$$

The optimal budget allocation between Γ_0 and S_2 is searched for by conducting the following thought experiment. An arbitrary allocation plan $(S_2, \Gamma_0 = M - S_2)$ is fixed. The principal may have to reallocate the resource between S_2 and Γ_0 to achieve the optimum. The direction of desirable reallocation thus completely depends on a comparison between the impact of the two instruments. In other words, when $\xi_2 = \left|\frac{2r-1}{\theta_1+(\theta_2+S_2)} - \frac{r}{\theta_2+S_2}\right| > \xi_3 = \left|\frac{-r}{V_0+M-S_2}\right|$, the resource must be directed away from the prize but towards S_2, while otherwise it goes the other way.

The trade-off between the subsidy and top-up prize is illustrated in Figure 1. The values for the parameters are as follows: $\theta_1 = 2$, $\theta_2 = 1$, $r = 1$, $V_0 = 1$, $M = 2$.

The downward-sloping solid curve plots $|\frac{\partial E(t)}{\partial S_2}|$, while the upward-sloping dotted curve plots $|\frac{\partial E(t)}{\partial \Gamma_0}|$. As the optimum requires a binding budget constraint of $S_2 + \Gamma_0 = M$, $\frac{\partial E(t)}{\partial \Gamma_0}$ is simply defined as functions of S_2 instead of Γ_0 itself. As shown in Figure 1, when S_2 increases, its marginal impact on $E(t)$ continues to decline, while it complementarily steps up the magnitude of $\frac{\partial E(t)}{\partial \Gamma_0}$. These observations tremendously facilitate the derivation of the following results.

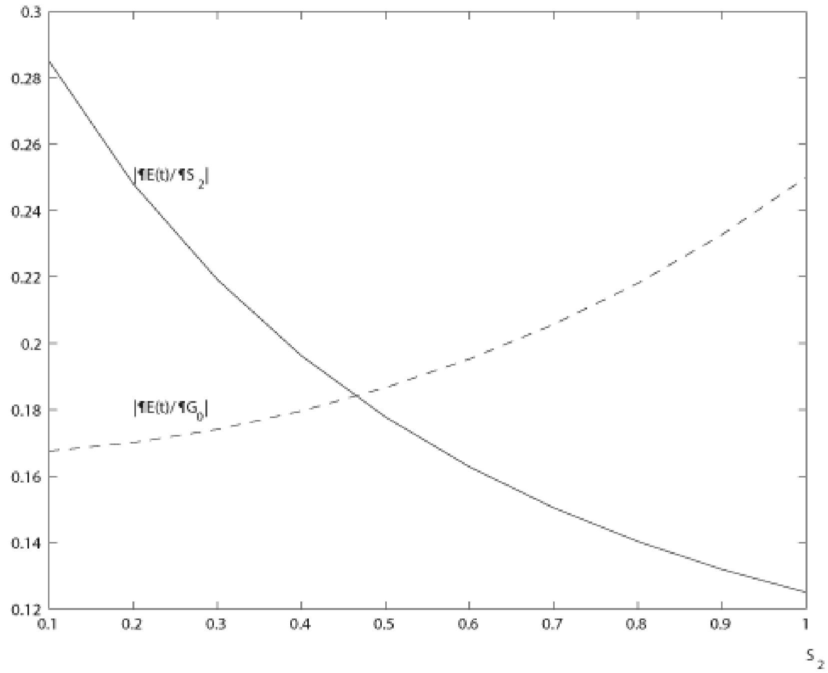

Figure 1: The Impacts of S_2 and Γ_0 on $E(t)$

Proposition 1 *When $\theta_1 - \theta_2 > M$, in the unique subgame perfect equilibrium, the principal*

(a) exhausts the entire budget to subsidize Firm 2, i.e., $S_2^* = M$, if and only if $V_0 \geq \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$;

(b) allocates the entire budget to the prize, i.e., $\Gamma_0^* = M$, if and only if $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$;

(c) splits its budget between the prize and the subsidy to Firm 2, if and only if

$$V_0 < \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)} \text{ and } V_0 + M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}. \tag{19}$$

In this case, the optimal $S_2^* \in (0, M)$ is the unique solution of

$$\frac{r\theta_1}{[\theta_1+(\theta_2+S_2)](\theta_2+S_2)} + \frac{1-r}{[\theta_1+(\theta_2+S_2)]} = \frac{r}{(V_0+M-S_2)}. \tag{20}$$

Proof. (a) Note $\frac{r\theta_1}{[\theta_1+(\theta_2+S_2)](\theta_2+S_2)} + \frac{1-r}{\theta_1+(\theta_2+S_2)}$ strictly decreases with S_2, and $\frac{r}{(V_0+M-S_2)}$ strictly increases with S_2. Thus it is optimal to have $S_2 = M$ if and only if $\frac{r\theta_1}{[\theta_1+(\theta_2+S_2)](\theta_2+S_2)} + \frac{1-r}{[\theta_1+(\theta_2+S_2)]} \geq \frac{r}{(V_0+M-S_2)}$ when $S_2 = M$. This requires

$$\begin{aligned}
\frac{r\theta_1}{[\theta_1+(\theta_2+M)](\theta_2+M)} + \frac{1-r}{[\theta_1+(\theta_2+M)]} &\geq \frac{r}{V_0} \Leftrightarrow \\
\frac{r\theta_1+(1-r)(\theta_2+M)}{[\theta_1+(\theta_2+M)](\theta_2+M)} &\geq \frac{r}{V_0} \Leftrightarrow \\
\frac{r[\theta_1+(\theta_2+M)](\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)} &\leq V_0.
\end{aligned}$$

(b) By the same argument as the proof for (a), it is optimal to allocate the entire budget to Γ_0 if and only if $\frac{r\theta_1}{[\theta_1+(\theta_2+S_2)](\theta_2+S_2)}+\frac{1-r}{[\theta_1+(\theta_2+S_2)]}\leq\frac{r}{V_0+M-S_2}$ when $S_2=0$, i.e. $\Gamma_0=M$. This requires

$$\begin{aligned}\frac{r\theta_1}{\theta_2(\theta_1+\theta_2)}+\frac{1-r}{(\theta_1+\theta_2)} &\leq \frac{r}{V_0+M}\Leftrightarrow\\ \frac{r\theta_1+(1-r)\theta_2}{\theta_2(\theta_1+\theta_2)} &\leq \frac{r}{V_0+M}\Leftrightarrow\\ V_0+M &\leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}.\end{aligned}$$

However, such an equilibrium of (b) may not exist at all. The above condition can hold only if $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}>M$.

(c) The results of (a) and (b) directly imply (c). When an interior solution ends up, the equilibrium would require (20) holds. Note that the left hand side of (20) strictly decreases with S_2, and the right side of (20) strictly increases with S_2. By intermediate value theorem, a unique solution $S_2^*\in(0,M)$ of (20) must exist if the conditions of Proposition 1(c) hold.

Q.E.D. ■

Proposition 1 states that the optimal budget allocation profile crucially depends on the private benefit V_0. More specifically, the principal should expend more resources subsidizing the ex ante weaker firm when the firms expect ample rewards from the patent, while it should expend more resources augmenting the prize purse when the patent value is insufficient. Simple algebra (Lemma 1) reveals that the impacts of additional subsidy S_2 and additional prize purse Γ_0 are both strictly decreasing. When the patent value falls within a medium range, an interior solution results and gives rise to both a positive subsidy and a top-up prize. Based on the arguments in the proof of Proposition 1(c), S_2^* and Γ_0^* are continuous at the thresholds $V_0=\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}-M$ and $V_0=\frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$.

What remains is to explicitly solve for the optimal subsidy S_2^*. Rearrange (19) and the following quadratic equation of S_2^* is obtained:

$$\begin{aligned}&S_2^{*2}+[r(2\theta_1+\theta_2)+r(V_0+M)+\theta_2-(V_0+M)]S_2^*\\&-[r(\theta_1-\theta_2)(V_0+M)-r\theta_2(\theta_1+\theta_2)+\theta_2(V_0+M)]=0.\end{aligned}$$

Corollary 1 *When $V_0<\frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$ and $V_0+M>\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$, the principal allocates an amount $S_2^*=\frac{\sqrt{A_1^2+4A_2}-A_1}{2}$ to subsidize the weaker firm, where $A_1\equiv[r(2\theta_1+\theta_2)+r(V_0+M)+\theta_2-(V_0+M)]$ and $A_2\equiv[r(\theta_1-\theta_2)(V_0+M)-r\theta_2(\theta_1+\theta_2)+\theta_2(V_0+M)]$.*

Proof. Two roots given by $\frac{-A_1 \pm \sqrt{A_1+4A_2}}{2}$, can be obtained by solving (20) using standard technique. By Proposition 1(c), a unique $S_2^* \in (0, M)$ exists. A_2 can be rewritten as

$$\begin{aligned} A_2 &= r\theta_2(\theta_1+\theta_2)(V_0+M)[\frac{r(\theta_1-\theta_2)}{r\theta_2(\theta_1+\theta_2)} - \frac{1}{(V_0+M)} + \frac{\theta_2}{r\theta_2(\theta_1+\theta_2)}] \\ &= r\theta_2(\theta_1+\theta_2)(V_0+M)[\frac{r\theta_1+(1-r)\theta_2}{r\theta_2(\theta_1+\theta_2)} - \frac{1}{(V_0+M)}]. \end{aligned}$$

$A_2 > 0$ by the condition $V_0 + M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. Thus, we must have $\sqrt{A_1^2+4A_2} > A_1$, and the unique positive root is $\frac{\sqrt{A_1^2+4A_2}-A_1}{2}$.

Q.E.D. ■

3.3 Case II: Mild Asymmetry ($\theta_1 - \theta_2 \in [0, M]$)

The case that involves mildly asymmetric firms is now considered. When $\theta_1 - \theta_2 \in [0, M]$, the principal has sufficient funds to fully counterbalance the asymmetry by preferentially subsidizing the weaker firm. However, a fully balanced playing field may not necessarily emerge in the optimum scenario, as a more generous prize purse stimulates effort supply as well.

Returning to the thought experiment conducted for the case of severe asymmetry, an arbitrary allocation plan is fixed and desirable reallocation is searched out. There are altogether two possible scenarios. In the first, a positive subsidy is provided to Firm 1, i.e., $S_1 > 0$. Lemma 2(b) thus requires $S_1 = S_2 - (\theta_1 - \theta_2)$ and $\Gamma_0 = M - (\theta_1 - \theta_2) - 2S_1$. That is, the principal fully offsets the initial imbalance. Note that the limiting situation $\Gamma_0 = 0$ is not excluded, while $(S_1 = 0, S_2 = \theta_1 - \theta_2)$ depicts the other limiting situation when $S_1 \to 0^+$. In the second scenario, no subsidy is provided to Firm 1, i.e., $S_1 = 0$. Lemma 1 thus implies that $S_2 \in [0, \theta_1 - \theta_2]$, where the asymmetry between firms continues to exist. Again, the limiting situation $\Gamma_0 = M$ is not excluded.

Assuming that the first scenario currently prevails, the optimum requires $S_1 = S_2 - (\theta_1 - \theta_2)$ and $\Gamma_0 = M - (\theta_1 - \theta_2) - 2S_1$. For all allocations that satisfy these conditions, the first order partial derivatives of (8) are given by

$$\frac{\partial E(t)}{\partial S_1} = \frac{\partial E(t)}{\partial S_2} = \frac{[(\theta_1+S_1)+(\theta_2+S_2)]^{2r-1}}{[r(\theta_2+S_2)(\theta_1+S_1)(V_0+\Gamma_0)]^r} \cdot \frac{-1}{\theta_1+\theta_2+S_1+S_2}, \tag{21}$$

$$\text{and } \frac{\partial E(t)}{\partial \Gamma_0} = \frac{[(\theta_1+S_1)+(\theta_2+S_2)]^{2r-1}}{[r(\theta_1+S_1)(\theta_2+S_2)(V_0+\Gamma_0)]^r} \cdot \frac{-r}{(V_0+M-S_1-S_2)}. \tag{22}$$

Again, the direction of a desirable reallocation depends on the magnitude of the RHS of (21) and (22). An additional subsidy is desirable if and only if $\frac{1}{\theta_1+\theta_2+S_1+S_2} > \frac{r}{V_0+M-S_1-S_2}$. By way of contrast, an additional prize purse is preferred if and only if $\frac{1}{\theta_1+\theta_2+S_1+S_2} < \frac{r}{V_0+M-S_1-S_2}$, and a desirable reallocation requires that S_1 and S_2 be reduced by an equal amount until the resource that subsidizes Firm 1 is completely taken away. When Firm 1 receives zero subsidy, the second scenario thus emerges, and the first order derivatives of $E(t)$ boil down to (17) and (18). Repeating the practice in Section 3.2.1 leads to the following results regarding the optimal allocation plan.

Proposition 2 *When $\theta_1 - \theta_2 \leq M$, in the unique subgame perfect equilibrium, the principal*

(a) allocates the entire budget to subsidize the two firms, i.e., $S_1^* = \frac{M-(\theta_1-\theta_2)}{2}, S_2^* = \frac{M+(\theta_1-\theta_2)}{2}$ and $\Gamma_0^* = 0$ if and only if $V_0 \geq r(\theta_1 + \theta_2 + M)$;

(b) subsidizes both firms and creates a positive prize, i.e., $S_1^* = S_2^* - (\theta_1 - \theta_2)$ and $\Gamma_0^* > 0$ if and only if $V_0 < r(\theta_1 + \theta_2 + M)$ and $V_0 + M > 2r\theta_1 + (\theta_1 - \theta_2)$;

(c) allocates the entire budget to the prize, i.e., $\Gamma_0^* = M$, if and only if $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$;

(d) splits the budget between the subsidy S_2 to the weaker firm and the prize, i.e., $S_1^* = 0, S_2^* \in (0, \theta_1 - \theta_2]$, and $\Gamma_0^* = M - S_2^* > M - (\theta_1 - \theta_2)$, if and only if $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0 + M < 2r\theta_1 + (\theta_1 - \theta_2)$.

Proof. (a) In scenario one, because $S_2 = S_1 + (\theta_1 - \theta_2)$, we have $\frac{1}{\theta_1+\theta_2+S_1+S_2}$ strictly decreases with S_1 while $\frac{r}{V_0+M-S_1-S_2}$ strictly increases with S_1. An optimum that involves zero Γ_0 would emerge if and only if $\frac{1}{\theta_1+\theta_2+S_1+S_2} \geq \frac{r}{V_0+M-S_1-S_2}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$. This condition can be written as

$$\frac{1}{\theta_1 + \theta_2 + M} \geq \frac{r}{V_0}, \tag{23}$$

which is equivalent to $V_0 \geq r(\theta_1 + \theta_2 + M)$.

Note that at allocation $(S_1 = 0, S_2 = \theta_1 - \theta_2)$, the comparison between (21) and (22) for scenario one is consistent with that between (17) and (18) for scenario two as the relevant derivatives take the same values (due to the continuity). Note that as long as the comparison between (17) and (18) for $(S_1 = 0, S_2 = \theta_1 - \theta_2)$ favors more research subsidies, then all allocations of scenario two are dominated by $(S_1 = 0, S_2 = \theta_1 - \theta_2)$.

The above arguments show that (23) is a sufficient condition for $\Gamma_0 = 0$. The necessity of this condition is clear as if it does not hold then deviating slightly from $\Gamma_0 = 0$ by creating positive top-up prize reduces the expected innovation time.

(b) Based on the arguments of case (a), to have the organizer subsidize both firms and create positive prizes, the necessary and sufficient condition is $\frac{1}{\theta_1+\theta_2+S_1+S_2} = \frac{r}{V_0+M-S_1-S_2}$ for some $S_1 \in (0, \frac{M-(\theta_1-\theta_2)}{2})$ and $S_2 = S_1 + (\theta_1 - \theta_2)$. This is equivalent to $\frac{1}{\theta_1+\theta_2+S_1+S_2} < \frac{r}{V_0+M-S_1-S_2}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$ and $\frac{1}{\theta_1+\theta_2+S_1+S_2} > \frac{r}{V_0+M-S_1-S_2}$ when $S_1 = 0$ and $S_2 = \theta_1 - \theta_2$. Condition $\frac{1}{\theta_1+\theta_2+S_1+S_2} < \frac{r}{V_0+M-S_1-S_2}$ when $S_1 = \frac{M-(\theta_1-\theta_2)}{2}$ and $S_2 = \frac{M+(\theta_1-\theta_2)}{2}$ means deviating slightly from $\Gamma_0 = 0$ by creating some top-up prize reduces the expected innovation time. It can be written as $V_0 < r(\theta_1+\theta_2+M)$. Condition $\frac{1}{\theta_1+\theta_2+S_1+S_2} > \frac{r}{V_0+M-S_1-S_2}$ when $S_1 = 0$ and $S_2 = \theta_1-\theta_2$ means that additional prize reduces the expected innovation time. The latter condition is satisfied if $V_0 + M > 2r\theta_1 + (\theta_1 - \theta_2)$. If $2r\theta_1 + (\theta_1 - \theta_2) \leq M$, this condition automatically holds.

(c) By the reasoning laid out above, an allocation that involves $S_2 < \theta_1 - \theta_2$ can be optimal if and only if $V_0 + M \leq 2r\theta_1 + (\theta_1 - \theta_2)$. Comparing (17) and (18) at $\Gamma_0 = M$ (i.e. $S_1 = S_2 = 0$) leads to that the optimal allocation plan involves zero subsidy and $\Gamma_0 = M$ if and only if $\frac{r\theta_1}{\theta_2(\theta_1+\theta_2)} + \frac{1-r}{(\theta_1+\theta_2)} \leq \frac{r}{V_0+M}$. We rewrite the inequality, and obtain $V_0 + M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. So far we found that such an optimal allocation plan of case (c) requires $V_0 + M$ be subject to two upper bounds, $2r\theta_1 + (\theta_1 - \theta_2)$ and $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$. We now compare the two upper bounds, and we claim the former is strictly greater than the latter. To see that, we have

$$\begin{aligned} & 2r\theta_1 + (\theta_1 - \theta_2) - \frac{r\theta_2(\theta_1 + \theta_2)}{r\theta_1 + (1-r)\theta_2} \\ = & \frac{[2r\theta_1 + (\theta_1 - \theta_2)][r\theta_1 + (1-r)\theta_2] - r\theta_2(\theta_1 + \theta_2)}{r\theta_1 + (1-r)\theta_2} \\ = & \frac{2r^2\theta_1(\theta_1 - \theta_2) + r\theta_1^2 + (1-r)\theta_2(\theta_1 - \theta_2)}{r\theta_1 + (1-r)\theta_2} > 0. \end{aligned} \tag{24}$$

(d) If the condition (c) (as well as those of (a) and (b)) is not satisfied, we must end up with an optimum with $S_1 = 0, S_2 \in (0, \theta_1 - \theta_2)$, and $\Gamma_0 > M - (\theta_1 - \theta_2)$ by the arguments we have laid out in the proof of Proposition 1(c).

Q.E.D. ■

Clearly, Proposition 2 and the arguments in its proof mean that S_1^*, S_2^* and Γ_0^* are continuous at thresholds $V_0 = \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$ and $V_0 = 2r\theta_1 + (\theta_1 - \theta_2) - M$. The amounts of equilibrium subsidies in the optimally designed contest are presented in the following corollary.

Corollary 2 *(a) When $V_0 < r(\theta_1 + \theta_2 + M)$ and $V_0 + M \geq 2r\theta_1 + (\theta_1 - \theta_2)$, the principal allocates subsidies $S_1^* = \frac{(V_0+M)-(1+2r)\theta_1+\theta_2}{2(1+r)}$ and $S_2^* = \frac{(V_0+M)+\theta_1-(1+2r)\theta_2}{2(1+r)}$, respectively, to Firm 1 and Firm 2.*

(b) When $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0 + M < 2r\theta_1 + (\theta_1 - \theta_2)$, the principal allocates a subsidy $S_2^* = \frac{\sqrt{A_1^2+4A_2}-A_1}{2}$ to Firm 2 only, where A_1 and A_2 are as defined in Corollary 1;

Proof. (a) When an interior equilibrium prevails which involves both firms receiving positive subsidies, proof of Proposition 2(b) shows that the equilibrium requires

$$\frac{1}{\theta_1 + \theta_2 + S_1^* + S_2^*} = \frac{r}{(V_0 + M - S_1^* - S_2^*)}. \tag{25}$$

Because $S_1^* = S_2^* + (\theta_1 - \theta_2)$, (25) can be rewritten as

$$\frac{1}{2(\theta_2 + S_2^*)} = \frac{r}{(V_0 + M + \theta_1 - \theta_2 - 2S_2^*)}. \tag{26}$$

We thus could obtain the optimal allocation bundle by solving this equation.

(b) In this case, the equilibrium condition is the same as that of Proposition 1(c). Equation (20) continues to apply and we obtain the desirable result.

Q.E.D. ■

When the private benefit is sufficiently small, i.e. $V_0 \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$, firms would not have sufficient incentive to conduct this research. Hence, all the money should take the form of a top-up prize in order to elicit the desired response. A greater private benefit strengthens firms' incentive to supply their effort, and therefore some money can be spared to subsidize the firms. When the patent value is sufficiently large, at the optimum the principal does not need to provide an additional prize incentive, while it prefers to subsidize the firms to increase their productivity.

Symmetric Firms: A Limiting Case

This model directly applies to the limiting case that involves symmetric firms, with $\theta_1 = \theta_2 = \theta$. In this case, the optimal allocation plan must either involve zero subsidy, or must equally subsidize the two firms. An allocation plan depicted by Proposition 2(d) could never emerge as the optimum. Proposition 2 is therefore adapted to obtain the following.

Corollary 3 *When $\theta_1 = \theta_2 = \theta$, in the unique subgame perfect equilibrium, the principal*

(a) allocates the entire budget to subsidize the two firms, i.e., $S_1^* = S_2^* = \frac{M}{2}$ and $\Gamma_0^* = 0$ if and only if $V_0 \geq r(2\theta + M)$;

(b) subsidizes both firms and creates a positive prize, i.e., $S_1^* = S_2^*$ and $\Gamma_0^* > 0$ if and only if $V_0 < r(2\theta + M)$, and $V_0 + M > 2r\theta$.

(c) allocates the entire budget to the prize, i.e., $\Gamma_0 = M$, if and only if $V_0 + M \leq 2r\theta$.

4 Discussion

The main results are first summarized in the following table.

Parameters	Optimal Contest
Case I: Severe Asymmetry $(\theta_1-\theta_2>M)$	
$V_0 \geq \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$	$S_1^*=0,\ S_2^*=M,\ \Gamma_0^*=0$
$V_0 < \frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}, V_0+M > \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^*=0,\ S_2^*>0,\ \Gamma_0^*>0$
$V_0+M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^*=0,\ S_2^*=0,\ \Gamma_0^*=M$
Case II: Mild Asymmetry $(\theta_1-\theta_2\in[0,M])$	
$V_0 \geq r(\theta_1+\theta_2+M)$	$S_1^*=S_2^*=\frac{M+(\theta_1-\theta_2)}{2}, \Gamma_0^*=0$
$V_0 < r(\theta_1+\theta_2+M), V_0+M > 2r\theta_1+(\theta_1-\theta_2)$	$S_1^*=S_2^*-(\theta_1-\theta_2)>0, \Gamma_0^*>0$
$\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} < V_0+M \leq 2r\theta_1+(\theta_1-\theta_2)$	$S_1^*=0, S_2^*>0, \Gamma_0^*>0$
$V_0+M \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$	$S_1^*=S_2^*=0, \Gamma_0^*=M$

Our analysis concerns itself with the optimal allocation of the budget M that minimizes innovation time. When more resources are available, the principal has additional flexibility in designing the incentive structure. This leads to the question of how the principal would reallocate its resources when it has a deeper pocket. In particular, would all the three elements in its bundle $(S_1^*, S_2^*, \Gamma_0^*)$ be assigned more resources?

Proposition 3 $S_1^*(M), S_2^*(M), \Gamma_0^*(M)$ *weakly increase with* M.

Proof. See Appendix. ■

Proposition 3 shows that all the three instruments are considered to be "normal goods" to the principal. When the resources available to the principal increases, they would never decrease the amount of resources allocated to any of the three instruments. Proposition 3 follows Lemmas 1 and 2 and further elaborates upon the roles played by these strategic instruments. In particular, it implies that an equilibrium involving a corner solution (i.e., zero resources on certain instruments) emerges more often if the resources are scarce, while a more balanced mix between subsidies and a prize would result when the amount of resources available increases. This result therefore reveals that subsidies and prizes are not perfectly substitutable in nature; although both are catalysts for success, they function through differing channels.

The rest of this section further examines the nature of the optimal contest structure and the role played by these structural elements (strategic instruments). In particular, we discuss how the value of the private benefit, the technological nature of the innovation project and firms' research capacities, would affect the optimal budget allocation profile.

4.1 The Private Benefit (Patent Value)

The optimal budget allocation profile depends critically on the patent value V_0. The results in both cases have consistently exhibited that the patent value V_0 is inversely related to the incidence of subsidies. The derivation of the result is straightforward. As implied by Lemma 1, when the winning firm can expect more rewards from a successful innovation, an additional prize would provide less incentive for further effort supply, while a subsidy that tends to amplify the output of these firms would increase its appeal.

An additional prize incentive is required only if the patent itself cannot adequately motivate these firms' innovative activities. As directly revealed by our results, an equilibrium where subsidies exhaust the entire budget could emerge if and only if the size of the private benefit is sufficiently large. By way of contrast, when V_0 is sufficiently low such that it falls below the threshold $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$, the contribution from an additional top-up prize completely outweighs that from subsidies, which leads to an equilibrium where no subsidy is given away. By a similar logic, the following result is expected.

Proposition 4 *The amounts of subsidies weakly increases with the patent value V_0.*

Proof. See Appendix. ■

Proposition 4 states that the patent value V_0 not only increases the frequencies of positive subsidies, but also increases the amount of resources allocated to subsidies in the equilibrium.

Our results thus directly shed light on the design of the incentive mechanism to motivate research efforts. Research firms would have weaker incentive to develop a dedicated military technology without a generous procurement contract awaiting the winner. This could explain the observation that DoD intensively seeks a prize incentive to motivate innovations that are often dedicated to military applications.

By way of contrast, governments and non-profit organizations such as the United Nations frequently dispense scarce resources in seeking medical cures or vaccines to limit the spread of deadly diseases. In contrast to dedicated military technology, the broader application of civil medical research leads to substantial profitability. Hence, in order to speed up the delivery of a medical discovery, priority could be more frequently given to research subsidies towards pharmaceutical research entities instead of luring them through procurement contracts.

4.2 The Technological Nature of the Innovation: The Role of "r"

The parameter r indicates the technological nature of the innovation. It literally measures the elasticity of the hazard rate to additional effort. Specifically, a greater r implies that the success of the innovation relies more heavily on continuing effort, rather than on a sudden spurt of inspiration. A lower r thus reflects that more uncertainty or greater difficulty is involved in a project.

A casual look at our results reveals that the properties of the optimal contest strongly depend on the magnitude of r. In general, the principal is more likely to allocate resources to the "top-up prize" (procurement contract) when the success of the project is more sensitive to additional effort, i.e., a greater r. This argument is demonstrated by analyzing the critical values of V_0 that define differing equilibria.

The case of severe asymmetry is first considered, i.e., $\theta_1 - \theta_2 > M$. An equilibrium where the subsidy S_2 exhausts the budget would emerge when V_0 exceeds the boundary $\frac{r(\theta_1+\theta_2+M)(\theta_2+M)}{r\theta_1+(1-r)(\theta_2+M)}$. Rewrite the boundary as $\frac{(\theta_1+\theta_2+M)(\theta_2+M)}{\theta_1+\frac{(1-r)}{r}(\theta_2+M)}$, and it can be seen to strictly increase with r. Thus, the condition that leads to zero top-up prize (Proposition 1(a)) is less likely to be met when r increases. In contrast, the top-up prize exhausts the budget (Proposition 1(b)) when $V_0 \leq \frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2} - M$. Again, a greater r lifts this bar, as $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$ strictly increases with r. The parameter r plays a similar role in the case of mild asymmetry, i.e., $\theta_1 - \theta_2 \leq M$. Thus, it can be concluded that when r increases, i.e., when the innovation is more sensitive to additional effort input, the incidence of positive subsidies would fall in response, while a prize incentive gains a greater appeal.

Proposition 5 *The equilibrium amounts of subsidies weakly decrease with r, i.e., the elasticity of the likelihood of success to additional effort.*

Proof. See Appendix. ■

Proposition 5 states that the amounts of subsidies increase with the level of uncertainty involved in the innovation project. The result of Proposition 5 is illustrated in Figure 2, which depicts in case one (severe asymmetry) the equilibrium subsidy to Firm two. The values for the parameters are as follows: $\theta_1 = 4$, $\theta_2 = 1$, $V_0 = 1$, $M = 1$. The curve that plots S_2^* is consistently shifted down when r rises from 0.8 to 1.0. Similar patterns can be observed when both S_1^* and S_2^* are plotted for other cases.

The implications of this result are directly revealed by the expression of equilibrium effort outlay as given by (7). The equilibrium effort outlay is linear in both the prize purse and the elasticity parameter r. A greater r amplifies the incentive provided by the winner prize, as it directly enlarges

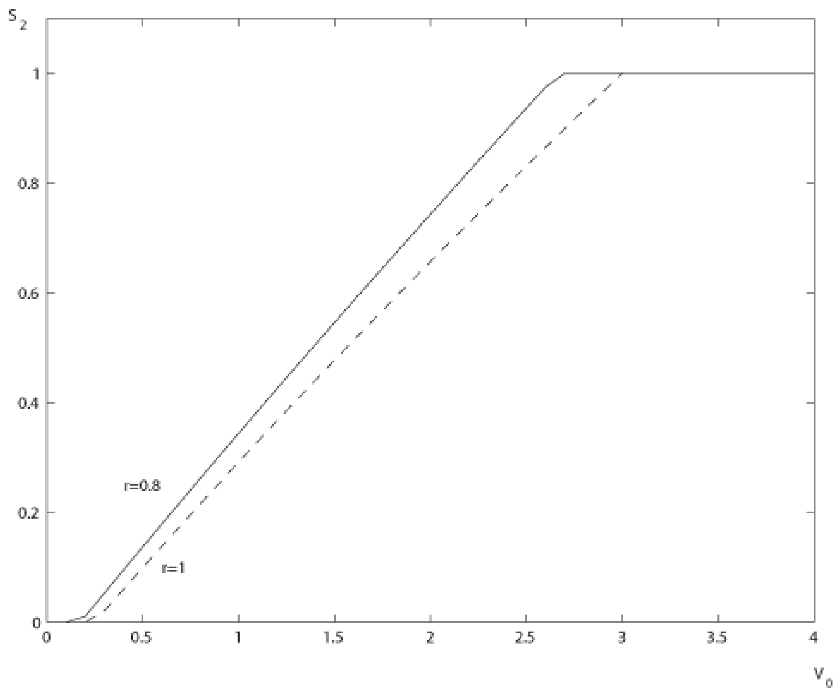

Figure 2: The Impact of r on S_2^*

the marginal impact of effort on winning chance. Thus, a greater r heightens the appeal of offering a prize incentive.

This result directly sheds light on the design of incentive mechanisms to promote scientific discovery. Intuitively, when competing parties hold out little prospects of success, a "prize" that is contingent on success cannot effectively motivate their effort. As a result, research subsidies that enhance the parties' capabilities would offer more compelling incentives when the project involves substantial difficulty or uncertainty. For example, a science foundation that aims to foster advancement in basic research usually provides incentives in the form of financial support to scientists more often than contingent monetary rewards. This is because breakthroughs in fundamental scientific theory presumably lead to immense reputational payoffs for the scientists who made the discovery. While this reputational benefit is a sufficient incentive to elicit effort, achieving the breakthrough is tremendously difficult. It is thus more attractive to provide a research grant that improves scientists' research capacity than to create an additional prize incentive.

4.3 Firms' Ex Ante Research Capacities

The results of this paper have confirmed the limited substitutability between a prize for winning firms and subsidies or aid to firms. As directly evidenced by the thought experiment in Section 3, a parallel increase in firms' efficiency parameters θ_is (which does not affect the balance in the competition) reduces the need to improve their capability. A more generous procurement contract would thus have a greater appeal. This substitutability therefore enables the principal to redirect its resources to the prize purse.

It has been shown that a subsidy to a firm increases (decreases) equilibrium effort outlays if and only if it balances (unbalances) the playing field. Thus, the principal has to preferentially subsidize the weaker firm when it balances the subsidies between firms.[18] This result echoes the logic espoused in the literature on efficient handicapping, such as the work of Che and Gale (1998, 2003) and Fu (2006). A more level playing field encourages the weaker firm to step up its efforts, which further prevents the stronger firm from slackening.

However, the following dichotomous effects should be stressed and clarified. Imagine that a positive exogenous shock affects firm 1. This allows its research capacity to increase by $\Delta\theta_1 > 0$, while further unbalancing the competition. For a given optimal resource allocation for (θ_1, θ_2, M), the shock to firm 1's capacity θ_1 exerts contrasting effects on the expected innovation time and effort supplied. While it strictly decreases the equilibrium amount of effort supplied, it still unambiguously increases the chances of success. This effect can be observed by fixing the resource allocation at the initial optimum. First, similar to (9), we have $\frac{\partial E(t)}{\partial \theta_1} = E(t)\left[\frac{2r-1}{\theta_1+\theta_2+S_1+S_2} - \frac{r}{\theta_1+S_1}\right] = \frac{\partial E(t)}{\partial S_1}$. Thus $\frac{\partial E(t)}{\partial \theta_1} < 0$ from Lemma 1. This indicates that the expected innovation time is shortened when the stronger firm realizes a favorable technological shock, holding constant the initial budget allocation plan.[19] Second, similar to (12), we have $\frac{\partial x^*}{\partial \theta_1} = \frac{r(V_0+\Gamma_0)(\theta_2+S_2)}{[(\theta_1+S_1)+(\theta_2+S_2)]^3} \cdot [(\theta_2+S_2) - (\theta_1+S_1)] \leq 0$ as $\theta_2 + S_2 \leq \theta_1 + S_1$ from Lemma 2. This reveals that when the stronger firm further improves its capability, it creates a negative incentive to both firms as an increasingly unbalanced playing field reduces the equilibrium amount of effort.[20] However, this negative effect is strictly dominated by the positive (direct) effect due to the capacity improvement of firms. As a result, the expected

[18] According to equation (15), a redistribution of the total amount of subsidies between the two firms does not have a direct impact on the expected time for developing an innovation. Instead, the redistribution of resources affects the expected innovation time through the equilibrium effort x^*.

[19] Clearly, the result still holds when the optimal resource allocation adjusts to the increase in θ_1.

[20] When the optimal resource allocation adjusts to the increase in θ_1, the impact on effort supplied needs to be studied more carefully.

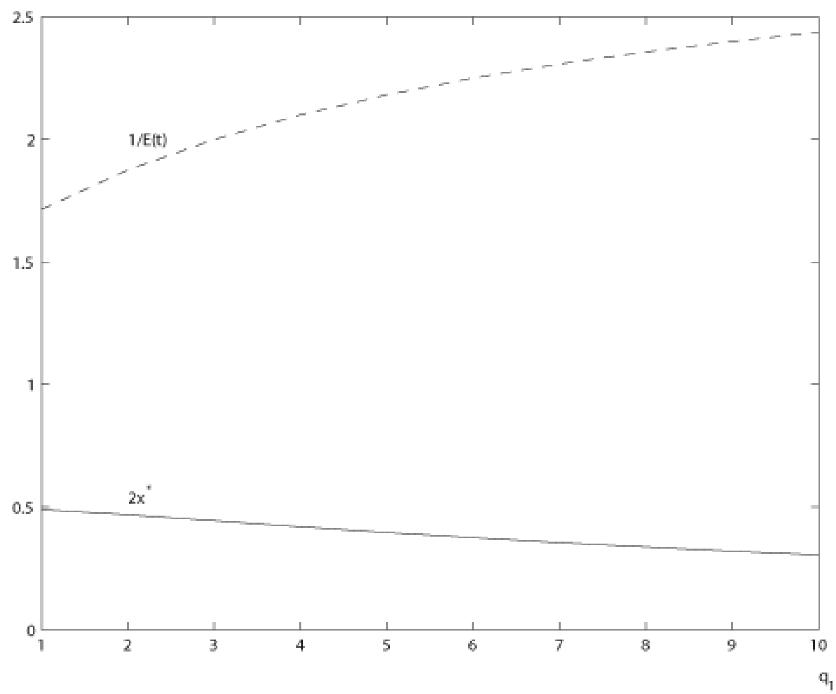

Figure 3: The Dichotomous Effects of θ_1 on $E(t)$ and total effort $2x^*$

innovation time strictly decreases. The principal would get better off even if the contest structure does not respond to this shock. These effects are illustrated in Figure 3. The values for the parameters are as follows: $V_0 = 1$, $\theta_2 = 1$, $r = 1$, $M = 2$.

This result has useful implications for the design of optimal contests. For example, the structure of an optimal contest will respond sensitively to the specific performance measure of interest. A principal's inclination for a stronger contestant 1 can immediately be reversed if he rather maximizes other performance measures, such as overall effort, which has been widely assumed in many studies. However, in this paper's setting, where the principal maximizes the overall output of competing firms, not their overall effort,[21] a stronger candidate always benefits the principal, although it could further unbalance the competition and reduce the amount of effort exerted.

4.4 Financially Constrained Firms

Financial constraints are not explicitly included in this model. Research activities are often impeded by limited resources. Most academics consider research funds to be inadequate to support scientific projects. As noted by Che and Gale (2003), DoD worries abut the financial stability of research

[21] Note that the expected innovation time is the inverse of the sum of firms' hazard rate functions.

companies, so it extensively subsidizes invited firms to maintain the competition. It appears natural to assume that firms are constrained by the resources available to them. However, it is worth noting that this model does not lose its bite in this regard. Recall that a firm's ex ante technology in our model is given by $h_i(x_i) = \theta_i x_i^r$. A binding financial constraint could function with the equivalence of a reduced capacity θ_i. Both a lower θ_i and a binding constraint tend to restrict the firms' actual equilibrium effort input. Hence, there is no loss of generality to presume that the generic measure θ_i of capacity reflects the firms' financial adequacy. Thus, when binding resource constraints are present, the results discussed in Section 3.2 directly allow us to predict increasing incidences of subsidies against a prize incentive.

4.5 When Firms Discount Future

In the current setting, it is explicitly assumed that firms do not discount future payoff. As pointed out in Section 2, this setting, although limited, could mirror a situation where the principal's interest is not well aligned with that of the firms, such that the desired innovation creates more imminent benefit to the principal than it does to the firms. If the firms also discount future payoff at a discount rate of $\widehat{\rho} \in (0, \rho)$, a firm i's expected payoff is thus described as

$$\begin{aligned} \pi_i(x_i, x_j) &= \int_0^\infty e^{-\widehat{\rho}t} f_i(t|x_i)(1 - F_j(t|x_j)) \cdot (V_0 + \Gamma_0)\, dt - x_i \\ &= \frac{h_i(x_i)}{h_i(x_i) + h_j(x_j) + \widehat{\rho}}(\Gamma_0 + V_0) - x_i. \end{aligned} \tag{27}$$

It is difficult to obtain close-form solutions in this setting, and this effect is not examined analytically. However, simple rationale reveals that a nonzero discount factor could affect the structure of the optimal contest. The discount rate $\widehat{\rho}$ plays a qualitatively opposite role to that of the elasticity parameter r. A greater $\widehat{\rho}$ diminishes the impact of any given prize incentive. As a result, the resource that is allocated to the top-up prize would be less effective in motivating these firms. Thus, returning to the results laid out in Section 3.2, when firms are more eager, the incidence and amounts of subsidies can be expected to increase. Alternatively, more eager firms implies that the interests of the principal and the firms are better aligned, which weakens the need for a prize incentive to elicit extra effort. Subsidies that amplify firms' given outputs thus have a greater appeal.

When $\widehat{\rho}$ remains moderate, the presence of this factor would not qualitatively vary the main results. However, additional complexity could result from a huge $\widehat{\rho}$. When $\widehat{\rho}$ is excessively large, it may no longer be optimal to preferentially subsidize the ex ante weaker firm. Looking at the payoff function (27), the discount factor can be interpreted as a third competitor in this race. When only two firms are engaged, preferentially subsidizing the weaker firm prevents the stronger one from slackening. However, this effect may disappear when $\widehat{\rho}$ is excessively large, i.e., when $\widehat{\rho} > h(x_2^*)$, as firm 2, the weakest competitor, cannot further motivate firm 1 even if it marginally steps up its effort. It could be more efficient, then, in that case to preferentially subsidize the stronger firm. The stronger firm exerts more productive effort, and an additional subsidy could further assist it to "race against the clock".

In summary, it should be noted that the above results apply in settings with moderately sized $\widehat{\rho}$. In order to view the panorama in the presence of a large $\widehat{\rho}$, additional analysis would be required, despite its analytical difficulty.

5 Concluding Remarks

This paper has studied the optimal design of research contests. The principal was allowed to design the contest using two strategic vehicles: subsidies to competing firms and a top-up prize (procurement contract). The principal faced a budget constraint and its objective was to minimize the expected amount of time required for an innovation. It was found that, when firms differ in their initial capabilities, the principal can effectively speed up the innovation by preferentially subsidizing the weaker firm. Furthermore, an additional prize incentive (procurement contract) occurs more often when less uncertainty is involved in the innovative project, while it occurs less frequently when there is a larger patent value (private benefit).

In order to gain more value from the results of this paper, future research should empirically assess the productivity of firms with asymmetric abilities, which is important for practical purposes when research grants are to be allocated between firms with heterogeneous capabilities. In addition, instead of assuming a constant marginal cost function, various cost functions can also be considered, to allow for more generality. To broaden the scope of our analysis and to further fit our setting into reality, contests that involve many participants and multiple non-identical prizes can also be calibrated as interesting extensions.

Appendix

5.1 Proof of Lemma 1

Proof. Since $r \leq 1$, we have $\frac{2r-1}{\theta_1+\theta_2+S_1+S_2} - \frac{r}{\theta_i+S_i} \leq \frac{r}{\theta_1+\theta_2+S_1+S_2} - \frac{r}{\theta_i+S_i} < 0$. Clearly, $\frac{\partial E(t)}{\partial \Gamma_0} < 0$. Now we show $\frac{\partial^2 E(t)}{\partial S_i^2} > 0$ and $\frac{\partial^2 E(t)}{\partial \Gamma_0^2} > 0$, i.e., their marginal impact decreases. To show $\frac{\partial^2 E(t)}{\partial S_1^2} > 0$, we need to show $\frac{\partial |\frac{\partial E(t)}{\partial S_1}|}{\partial S_1} < 0$. Because $|\frac{\partial E(t)}{\partial S_1}| = E(t)\,\xi_1$, it suffices to show ξ_1 decreases with S_1. Simple calculus reveals that $\frac{\partial \xi_1}{\partial S_1} = \frac{2r-1}{(\theta_1+\theta_2+S_1+S_2)^2} - \frac{r}{(\theta_1+S_1)^2} \leq \frac{r}{(\theta_1+\theta_2+S_1+S_2)^2} - \frac{r}{(\theta_1+S_1)^2} < 0$. The marginal impacts of S_2 and Γ_0 can be similarly shown.

Q.E.D. ■

5.2 Proof of Proposition 3

Proof. We first consider the case of severe asymmetry ($\theta_1 - \theta_2 > M$) where S_1^* must be zero. Clearly, the claim is true for the case of Proposition 1(a) and 1(b). For case 1(c), the equilibrium subsidy S_2^* is uniquely determined by the equation $\frac{r\theta_1}{[\theta_1+(\theta_2+S_2)](\theta_2+S_2)} + \frac{1-r}{[\theta_1+(\theta_2+S_2)]} = \frac{r}{(V_0+M-S_2)}$. When V_0 increases, RHS would decrease in response, which requires a larger S_2^* to rebalance the equation.

We then consider the case of mild asymmetry ($\theta_1 - \theta_2 \leq M$). A similar logic applies in equilibrium with only $S_2^* > 0$.

We then consider the equilibrium that involves positive subsidies to both firms. The equilibrium amounts of S_2^*, as well as $S_1^* = S_2^* - (\theta_1 - \theta_2)$ are uniquely determined by the equation $\frac{1}{2(\theta_2+S_2^*)} = \frac{r}{(V_0+M+\theta_1-\theta_2-2S_2^*)}$. When V_0 increases, RHS strictly decreases, which thus requires an increase in S_2^* to restore the balance.

An increasing V_0 could vary the type of the equilibrium. However, by the proofs of Proposition 1 and 2, S_1^* and S_2^* are continuous at thresholds. In addition, as pointed out in the text, an increasing V_0 increases the likelihood of the equilibrium with the entire budget to be allocated to subsidies. We then conclude that S_i^* increases with V_0.

Q.E.D. ■

5.3 Proof of Proposition 4

Proof. Suppose that we have $M_1 > M_0$. We claim $S_1^*(M_0) \leq S_1^*(M_1), S_2^*(M_0) \leq S_2^*(M_1)$, and $\Gamma_0^*(M_0) \leq \Gamma_0^*(M_1)$. Define $\Delta \equiv M_1 - M_0 > 0$. Initially the principal has an optimum

$(S_1^*(M_0), S_2^*(M_0), \Gamma_0^*(M_0))$. Note that we must have $S_2^*(M_0) \geq S_1^*(M_0)$ from Lemma 2. We consider the following possible cases.

Case 1: $\Gamma_0^*(M_0) = 0$**,** $S_1^*(M_0) = 0, S_2^*(M_0) = M_0$**.**

In this case we must have $\xi_2^*(M_0) \geq \max\{\xi_1^*(M_0), \xi_3^*(M_0)\}$. Recall that $\xi_1 = \frac{r}{\theta_1+S_1} - \frac{2r-1}{\theta_1+\theta_2+S_1+S_2}$, $\xi_2 = \frac{r}{\theta_2+S_2} - \frac{2r-1}{\theta_1+\theta_2+S_1+S_2}$, and $\xi_3 = \frac{r}{V_0+\Gamma_0}$. In order to verify the claim, we only need to show that $S_2^*(M_0) \leq S_2^*(M_1)$.

Suppose $S_2^*(M_0) > S_2^*(M_1)$, then we must have $S_1^*(M_0) < S_1^*(M_1)$ or $\Gamma_0^*(M_0) < \Gamma_0^*(M_1)$. By Lemma 2, $S_1^*(M_1)$ must remain at zero if $S_2^*(M_0) > S_2^*(M_1)$. Thus, we can only have $\Gamma_0^*(M_1) > \Gamma_0^*(M_0) = 0$. However, this implies $\xi_2^*(M_1) > \xi_3^*(M_1)$, which conflicts with the first order conditions for the optimum.

Case 2: $\Gamma_0^*(M_0) = 0$**,** $S_1^*(M_0) \in (0, M_0), S_2^*(M_0) \in (0, M_0)$**.**

In this case we must have $\xi_1^*(M_0) = \xi_2^*(M_0) \geq \xi_3^*(M_0)$. In this case, we need to show neither S_1^* nor S_2^* drops. By Lemma 2, when one of S_i^* drops, the other must follow. Hence, suppose that both S_1^* and S_2^* drop, which implies Γ_0^* must strictly increase. ξ_3^* then strictly decrease, but at least ξ_2^* increases since $\theta_1 + S_1^* \geq \theta_2 + S_2^*$ from Lemma 2. To see that, we have $\xi_2^*(M_1) = \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)} + \frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)} \geq \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r}{2[\theta_2+S_2^*(M_1)]} + \frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)} = \frac{1}{\theta_1+S_1^*(M_1)+\theta_2+S_2^*(M_1)}$. $\xi_2^*(M_0) = \frac{r}{\theta_2+S_2^*(M_0)} - \frac{2r}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)} + \frac{1}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)} = \frac{1}{\theta_1+S_1^*(M_0)+\theta_2+S_2^*(M_0)}$ since $\theta_1 + S_1^*(M_0) = \theta_2 + S_2^*(M_0)$. We thus have $\xi_2^*(M_1) > \xi_2^*(M_0) \geq \xi_3^*(M_0) > \xi_3^*(M_1)$. This means that the resources on $\Gamma_0^*(M_1)$ instead should be reallocated to S_2^*.

Case 3: $\Gamma_0^*(M_0) \in (0, M_0)$**,** $S_1^*(M_0) = 0, S_2^*(M_0) \in (0, M_0)$**.**

In this case we must have $\xi_2^*(M_0) = \xi_3^*(M_0) \geq \xi_1^*(M_0)$. We need to show neither S_2^* nor Γ_0^* drops. Suppose that S_2^* drops, we must have $S_1^*(M_1) = 0$ by Lemma 2. This means Γ_0^* must increase. Hence, ξ_3^* decreases while ξ_2^* increases, which yields $\xi_2^* > \xi_3^*$. Thus, the resources on $\Gamma_0^*(M_1)$ instead should be reallocated to S_2^*.

Suppose Γ_0^* decreases, we must have the total resources on S_1^* and S_2^* increases. In this case, we must have S_2^* increases. Note $\xi_2^*(M_1) = \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r-1}{\theta_1+\theta_2+S_1^*(M_1)+S_2^*(M_1)}$. Consider two cases.

First, if $S_1^*(M_1) = 0$, then $\xi_2^*(M_1) = \frac{r}{\theta_2+S_2^*(M_1)} - \frac{2r-1}{\theta_1+\theta_2+S_2^*(M_1)}$, which implies $\xi_2^*(M_1) < \xi_2^*(M_0)$ as S_2^* has increased. Second, $S_1^*(M_1) > 0$. Because $\theta_1 + S_1^*(M_1) = \theta_2 + S_2^*(M_1)$, $\xi_2^*(M_1) = \frac{1}{2(\theta_2+S_2^*(M_1))}$. It also implies $\xi_2^*(M_1) < \xi_2^*(M_0)$ because $\xi_2^*(M_0) = \frac{r}{\theta_2+S_2^*(M_0)} - \frac{2r-1}{\theta_1+\theta_2+S_2^*(M_0)} = \frac{r}{\theta_2+S_2^*(M_0)} - \frac{2r}{\theta_1+\theta_2+S_2^*(M_0)} + \frac{1}{\theta_1+\theta_2+S_2^*(M_0)} > \frac{1}{\theta_1+\theta_2+S_2^*(M_0)} \geq \frac{1}{2(\theta_2+S_2^*(M_0))}$.

Thus eventually, ξ_3^* increases while ξ_2^* decreases. This means the resources on S_2^* instead should be reallocated to $\Gamma_0^*(M_1)$.

Case 4: $\Gamma_0^*(M_0) \in (0, M_0)$**,** $S_1^*(M_0) \in (0, M_0), S_2^*(M_0) \in (0, M_0)$**.**

In this case we must have $\xi_1^*(M_0) = \xi_2^*(M_0) = \xi_3^*(M_0)$. We need to show none of the three choice variables can decrease. Combining the arguments in Cases 2 and 3 leads to this result.

Case 5: $\Gamma_0^*(M_0) = M_0$**,** $S_1^*(M_0) = 0, S_2^*(M_0) = 0$**.**

In this case we must have $\xi_3^*(M_0) \geq \max\{\xi_1^*(M_0), \xi_2^*(M_0)\}$. We need to show Γ_0^* cannot drop. Similar arguments as in Case 3 would apply.

Q.E.D. ■

5.3.1 Proof of Proposition 5

Proof. We consider the impact of a marginal increase in r on the equilibrium in two possible cases

.

Case 1: $\theta_1 - \theta_2 > M$**.**

In this case, $S_1^* = 0$. We claim that in any equilibrium where $S_2^* > 0$, S_2^* strictly decreases with r. The equilibrium condition (20) can be rewritten as

$$\frac{\theta_1}{[\theta_1 + (\theta_2 + S_2^*)](\theta_2 + S_2^*)} + \frac{1-r}{r[\theta_1 + (\theta_2 + S_2^*)]} = \frac{1}{(V_0 + M - S_2^*)}. \tag{28}$$

Assume there is an equilibrium with $S_2^* > 0, \Gamma_0^* > 0$. We fix this equilibrium and hold S_2^* constant. For an increase in r, the LHS of (28) would strictly decrease, while the RHS remains constant. To restore the balance, S_2 must be reduced.

Case 2: $\theta_1 - \theta_2 \leq M$**.**

Consider equation (26). Assume there is an equilibrium with $S_1^*, S_2^*, \Gamma_0^* > 0$ and we fix this equilibrium and hold S_2^* (as well as S_1^* since $S_1^* = S_2^* - (\theta_1 - \theta_2)$) constant. Imagine a marginal increase in r. We would see that the RHS of (26) would strictly increase, while RHS remains constant. To restore the balance, S_2^* (and S_1^* since $S_1 = S_2 - (\theta_1 - \theta_2)$) must be reduced.

By the same argument as we presented in case 1, this result holds in an equilibrium with $S_1^* = 0$ and $S_2^*, \Gamma_0^* > 0$.

In the reasoning we lay out above, we implicitly assume that the increase in r is marginal such that it does not cause a differing type of equilibrium. As aforementioned, an increase in r raises all the cutoffs for V_0 for different types of equilibria. Thus, the claim continues to hold when the change in r is not a marginal one. In case 1, assume that an increase in r causes $V_0 + M$ to fall below $\frac{r\theta_2(\theta_1+\theta_2)}{r\theta_1+(1-r)\theta_2}$, then S_2^* would drop to zero (Proposition 1). In case 2, the same would happen to S_1^* when an increase in r causes $V_0 + M$ falls below $2r\theta_1 + (\theta_1 - \theta_2)$ (Proposition 2). Such a change would cause S_2^* to decline as well. This can be seen by the proof laid out above and the fact that S_2^* is continuous on $V_0 + M$.

Q.E.D. ■

References

[1] Baye, M., and Hoppe, H., The Strategic Equivalence of Rent-Seeking, Innovation, and Patent-Race Games, *Games and Economic Behavior*, 2003, 44, 217-226.

[2] Che, Y.K., and Gale, I., Caps on Political Lobbying, *American Economic Review*, 88, 1998, 646-671.

[3] Che, Y.K., and Gale, I., Optimal Design of Research Contests, *American Economic Review*, 93, 2003, 643-651.

[4] Day, John R., *Trains*. New York: Bantam Books, 1971.

[5] Denecolò, V., Two-Stage Patent Races and Patent Policy, Rand Journal of Economics, 2000, 31, 488-501.

[6] Denecolò, V., and Franzoni, L.A., On the Winner-Take-All Principle in Innovation Races, 2007, mimeo.

[7] Dasgupta, Partha & Stiglitz, Joseph, Industrial Structure and the Nature of Innovative Activity, *Economic Journal*, 1980, 90, 266-93.

[8] Etro, F., Innovation by Leaders, Economic Journal, 2004, 114, 281-303.

[9] Fang, H., Lottery versus All Pay Auction Models of Lobbying, *Public Choice*, 2002, 351-371

[10] Fu, Q., A Theory of Affirmative Action in College Admissions, *Economic Inquiry*, 2006, 44, 420-429, 2006.

[11] Fu, Q., and Lu, J., 2007a. The Optimal Multiple-Stage Contest, *NAJ Economics*, 14.

[12] Fu, Q., and Lu. J., 2007b. Unifying Contests: from Noisy Ranking to Ratio-Form Contest Success Functions, working paper.

[13] Fu, Q., and Lu, J., Can It be Too Crowded in Contests: Over Competition with Pre-Investment, Mimeo, 2008.

[14] Fullerton, R.L., and McAfee, P.R., Auctioning Entry into Tournaments, *Journal of Political Economy*, 1999, 107, 573-605.

[15] Gonzáez, X., Jaumandreu, J. & Pazo, C., Barriers to Innovation and Subsidy Effectiveness, *Rand Journal of Economics*, 36, 2005, 930-50.

[16] Gradstein, M., and Konrad, K.A., Orchestrating Rent Seeking Contests, *Economic Journal*, 1999, 109, 535-545.

[17] Kaplan, T., Luski, I., and Wettstein, D., Innovative Activity with Sunk Cost, International Journal of Industrial Organization , 2003, 21, 1111-1133.

[18] Konrad, K. A. (2007) "Strategy in Contests — an Introduction" Discussion Paper SP II 2007-01, Wissenschaftszentrum, Berlin.

[19] Kräkel M., R&D Spillovers and Strategic Delegation in Oligopolistic Contests, *Managerial and Decision Economics*, 2004, 25, 147-156.

[20] Lach, S. (2002) "Do R&D subsidies stimulate or displace private R&D? Evidence from Israel." *Journal of Industrial Economics*, 50: 369–90.

[21] Lee, T., and Wilde, L., Market Structure and Innovation: A Reformulation, *Quarterly Journal of Economics*, 1980, 94, 429-436.

[22] Lichtenberg F.R., The Private R and D Investment Response to Federal Design and technical Competitions, *American Economic Review*, 1988, 78, 550-559.

[23] Matros, A., Elimination Tournaments where Players Have Fixed Resources, 2005, working paper, University of Pittsburgh.

[24] Moldovanu, B., and Sela, A., The Optimal Allocation of Prizes in Contests, *American Economic Review*, 2001, 91, 542-558.

[25] Münster J., Contests with Investment, *Managerial and Decision Economics*, Vol. 28, Issue 8, 2007, 849-862.

[26] Reinganum, J., The Timing of Innovation: Research, Development, and Diffusion, In R. Schmalensee and R.D. Willig, eds, *The Handbook of Industrial Organization*, New York: North-Holland, 1989.

[27] Rosen, S., Prizes and Incentives in Elimination Tournaments, *American Economic Review*, 1986, 76, 701-715.

[28] Taylor C.R., Digging for Golden Carrots: An Analysis of Research Tournaments, *The American Economic Review*, 1995, 85 (4), 872-890.